

# Bidirectional Type Checking for Existential Types with Higher-Rank Polymorphism

Who? Hasti Toossi, Ningning Xie

From? University of Toronto

When? ESOP 2026

Types

 $Vec\ n\ a$ 

Example

 $filter :: \forall n\ a. (a \rightarrow Bool) \rightarrow Vec\ n\ a \rightarrow \exists m. Vec\ m\ a$  $filter = \lambda p\ vec \rightarrow \mathbf{case\ } vec\ \mathbf{of}$  $Nil \rightarrow Nil$  $(:>) x\ xs$  $| p\ x \rightarrow (:>) x\ (filter\ p\ xs)$  $| otherwise \rightarrow filter\ p\ xs$

Types

*pack*  $t, e$  *as*  $\exists a. t'$   
*unpack*  $e$  *as*  $t, x$  **in**  $e'$

Example

*filter*  $:: \forall n a. (a \rightarrow \text{Bool}) \rightarrow \text{Vec } n a \rightarrow \exists m. \text{Vec } m a$   
*filter* =  $\Lambda n a. \lambda(p :: a \rightarrow \text{Bool}) (\text{vec} :: \text{Vec } n a) \rightarrow$   
**case** *vec* **of**  
*Nil*  $\rightarrow$  *pack* *Zero*, *Nil* *as*  $\exists m. \text{Vec } m a$   
 $(: >)$   $n a (x :: a) (xs :: \text{Vec } n a)$   
 |  $p x \rightarrow$  *unpack* (*filter*  $n a p xs$ ) *as*  $n1, ys$  **in**  
   *pack* *Succ*  $n1, (: >)$   $n1 a x ys$  *as*  $\exists m. \text{Vec } m a$   
 | *otherwise*  $\rightarrow$  *filter*  $n a p xs$

Types

$$\text{pack } t, e \text{ as } \exists a. t'$$

$$[e]$$

Expressions

$$\text{open } e$$

Example

$$\text{filter} :: \forall n a. (a \rightarrow \text{Bool}) \rightarrow \text{Vec } n a \rightarrow \exists m. \text{Vec } m a$$

$$\text{filter} = \Lambda n a. \lambda(p :: a \rightarrow \text{Bool}) (\text{vec} :: \text{Vec } n a) \rightarrow$$

**case**  $\text{vec}$  **of**

- $\text{Nil} \rightarrow \text{pack } \text{Zero}, \text{Nil} \text{ as } \exists m. \text{Vec } m a$
- $(:>) n a (x :: a) (xs :: \text{Vec } n a)$ 
  - $| p x \rightarrow \text{let } ys = \text{filter } n a p xs \text{ in}$
  - $\text{pack } \text{Succ } [ys], (:>) [ys] a x (\text{open } ys)$
  - $\text{as } \exists m. \text{Vec } m a$
- $| \text{otherwise} \rightarrow \text{filter } n1 a p xs$

Types

 $Vec\ n\ a$ 

Example

 $filter :: \forall n\ a. (a \rightarrow Bool) \rightarrow Vec\ n\ a \rightarrow \exists m. Vec\ m\ a$  $filter = \lambda p\ vec \rightarrow \mathbf{case\ } vec\ \mathbf{of}$  $Nil \rightarrow Nil$  $(:>) x\ xs$  $| p\ x \rightarrow (:>) x\ (filter\ p\ xs)$  $| otherwise \rightarrow filter\ p\ xs$

EDWL  
ICFP 2021

## An Existential Crisis Resolved

Type Inference for First-Class Existential Types

**RICHARD A. EISENBERG**, Tweag, France

**GUILLAUME DUBOC**, ENS Lyon, France and Tweag, France

**STEPHANIE WEIRICH**, University of Pennsylvania, USA

**DANIEL LEE**, University of Pennsylvania, USA

## EDWL

- A **declarative** type system
  - strong existential types
  - higher-rank polymorphism
- An **elaboration** of the decl. system
  - type soundness

EDWL  
ICFP 2021

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Quote

*“A real implementation might use unification variables, but we here rely on the rich body of literature [e.g. Dunfield and Krishnaswami 2013] that allows us to guess monotypes during type inference, knowing how to translate this convention into an implementation using unification variables.”*

## This Work

- A **declarative** type system
  - strong existential types
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  - subtyping

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  - **unification** and **promotion**
  - **sound but incomplete w.r.t. the decl. system**
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## This Work

- A **declarative** type system
  - strong existential types
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  - **unification** and **promotion**
  - **sound but incomplete w.r.t. the decl. system**
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- An **elaboration** of the decl. system
  - type soundness
- **Haskell implementation**

Types

$$\exists a. \epsilon$$

$$[e : \exists a. \epsilon]$$

Key Rule

D-I-ABS

$$\frac{\begin{array}{l} \Gamma, x : \tau \vdash e \Rightarrow \sigma \\ \Gamma, x : \tau \vdash \sigma \rightsquigarrow_{\forall} \epsilon \\ \bar{a} \text{ fresh} \quad \epsilon' = [\bar{a}/[\epsilon]_x]\epsilon \end{array}}{\Gamma \vdash \lambda x. e \Rightarrow \tau \rightarrow \exists \bar{a}. \epsilon'}$$

Example

- $\vdash \lambda x : \text{Int}. \lambda y : [x]. y \Rightarrow ?$

## Example

•  $\vdash \lambda x : \text{Int}. \lambda y : [x]. y \Rightarrow ?$

1  $x : \text{Int} \vdash \lambda y : [x]. y \Rightarrow [x] \rightarrow [x]$

Example

- $\vdash \lambda x : \text{Int}. \lambda y : [x]. y \Rightarrow \text{Int} \rightarrow [x] \rightarrow [x]$
- 1  $x : \text{Int} \vdash \lambda y : [x]. y \Rightarrow [x] \rightarrow [x]$

## Example

- $\vdash \lambda x : \text{Int}. \lambda y : [x]. y \Rightarrow \text{Int} \rightarrow \exists a. a \rightarrow a$
- 1  $x : \text{Int} \vdash \lambda y : [x]. y \Rightarrow [x] \rightarrow [x]$

Example

- $\vdash \lambda x : \text{Int}. \lambda y : [x]. y \Rightarrow \text{Int} \rightarrow \exists a. a \rightarrow a$

- 1  $x : \text{Int} \vdash \lambda y : [x]. y \Rightarrow [x] \rightarrow [x]$

Example

Let  $e = \lambda x : \text{Int}. \lambda y : [x]. y$ .

- $\vdash e \# e' \Rightarrow ?$

Example

- $\vdash \lambda x : \text{Int}. \lambda y : [x]. y \Rightarrow \text{Int} \rightarrow \exists a. a \rightarrow a$

1  $x : \text{Int} \vdash \lambda y : [x]. y \Rightarrow [x] \rightarrow [x]$

Example

Let  $e = \lambda x : \text{Int}. \lambda y : [x]. y$ .

- $\vdash e1 e' \Rightarrow ?$

1 •  $\vdash e1 \Rightarrow \exists a. a \rightarrow a$

Example

- $\vdash \lambda x : \text{Int}. \lambda y : [x]. y \Rightarrow \text{Int} \rightarrow \exists a. a \rightarrow a$

- 1  $x : \text{Int} \vdash \lambda y : [x]. y \Rightarrow [x] \rightarrow [x]$

Example

Let  $e = \lambda x : \text{Int}. \lambda y : [x]. y$ .

- $\vdash e1e' \Rightarrow ?$

- 1 •  $\vdash e1 \Rightarrow \exists a. a \rightarrow a$

- 2 Instantiate  $\exists a. a \rightarrow a$  to

$$[e1 : \exists a. a \rightarrow a] \rightarrow [e1 : \exists a. a \rightarrow a].$$

Example

- $\vdash \lambda x : \text{Int}. \lambda y : [x]. y \Rightarrow \text{Int} \rightarrow \exists a. a \rightarrow a$

- 1  $x : \text{Int} \vdash \lambda y : [x]. y \Rightarrow [x] \rightarrow [x]$

Example

Let  $e = \lambda x : \text{Int}. \lambda y : [x]. y$ .

- $\vdash e1 e' \Rightarrow [e1 : \exists a. a \rightarrow a]$

- 1 •  $\vdash e1 \Rightarrow \exists a. a \rightarrow a$

- 2 Instantiate  $\exists a. a \rightarrow a$  to

$$[e1 : \exists a. a \rightarrow a] \rightarrow [e1 : \exists a. a \rightarrow a].$$

- 3 •  $\vdash e' \Leftarrow [e1 : \exists a. a \rightarrow a]$

DK  
ICFP 2013

## Complete and Easy Bidirectional Typechecking for Higher-Rank Polymorphism

Jana Dunfield    Neelakantan R. Krishnaswami

Max Planck Institute for Software Systems  
Kaiserslautern and Saarbrücken, Germany  
jd169@queensu.ca    nk480@cl.cam.ac.uk

Types

 $\hat{a}$ 

Contexts

 $\Delta, \hat{a}$  $\Delta, \hat{a} = \tau$ 

Judgements

 $\Delta_1 \vdash e \Rightarrow \sigma \dashv \Delta_2$

## Example

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow ?$

1  $x : \text{Int} \vdash \lambda y. y \Rightarrow ?$

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow ?$

1  $x : \text{Int} \vdash \lambda y. y \Rightarrow ?$

## Example

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow \text{Int} \rightarrow \hat{a} \rightarrow \hat{a} \dashv \hat{a}$

1  $x : \text{Int} \vdash \lambda y. y \Rightarrow \hat{a} \rightarrow \hat{a} \dashv \hat{a}$

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow ?$

1  $x : \text{Int} \vdash \lambda y. y \Rightarrow \tau \rightarrow \tau$

## Example

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow \text{Int} \rightarrow \hat{a} \rightarrow \hat{a} \dashv \hat{a}$

1  $x : \text{Int} \vdash \lambda y. y \Rightarrow \hat{a} \rightarrow \hat{a} \dashv \hat{a}$

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow ?$

1  $x : \text{Int} \vdash \lambda y. y \Rightarrow \tau \rightarrow \tau$

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow \text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$

1  $x : \text{Int} \vdash \lambda y. y \Rightarrow \text{Int} \rightarrow \text{Int}$

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow \text{Int} \rightarrow (\text{Int} \rightarrow \text{Int}) \rightarrow (\text{Int} \rightarrow \text{Int})$

1  $x : \text{Int} \vdash \lambda y. y \Rightarrow (\text{Int} \rightarrow \text{Int}) \rightarrow (\text{Int} \rightarrow \text{Int})$

## Soundness

If the **algorithmic** system infers type  $\sigma'$  for  $e$ , the **declarative** system can infer any solution  $\sigma$  of  $\sigma'$ .

## Completeness?

If the **declarative** system can infer type  $\sigma$  for  $e$ , the **algorithmic** system infers  $\sigma'$  such that  $\sigma$  is a solution of  $\sigma'$ .

## Example

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow \text{Int} \rightarrow \hat{a} \rightarrow \hat{a} \vdash \hat{a}$

**1**  $x : \text{Int} \vdash \lambda y. y \Rightarrow \hat{a} \rightarrow \hat{a} \vdash \hat{a}$

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow ?$

**1**  $x : \text{Int} \vdash \lambda y. y \Rightarrow \tau \rightarrow \tau$

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow \text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$

**1**  $x : \text{Int} \vdash \lambda y. y \Rightarrow \text{Int} \rightarrow \text{Int}$

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow \text{Int} \rightarrow (\text{Int} \rightarrow \text{Int}) \rightarrow (\text{Int} \rightarrow \text{Int})$

**1**  $x : \text{Int} \vdash \lambda y. y \Rightarrow (\text{Int} \rightarrow \text{Int}) \rightarrow (\text{Int} \rightarrow \text{Int})$

•  $\vdash \lambda x : \text{Int}. \lambda y. y \Rightarrow \text{Int} \rightarrow \exists a. a \rightarrow a$

**1**  $x : \text{Int} \vdash \lambda y. y \Rightarrow [x] \rightarrow [x]$

Table

Types	Is $\sigma'$ a solution of $\sigma$ ?
$\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$ $\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$	yes
$\text{Int} \rightarrow \exists a.a \rightarrow a$ $\text{Int} \rightarrow \exists a.a \rightarrow a$	yes
$\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$ $\text{Int} \rightarrow \hat{a} \rightarrow \hat{a}$	yes
$\text{Int} \rightarrow (\text{Int} \rightarrow a) \rightarrow (\text{Int} \rightarrow a)$ $\text{Int} \rightarrow \hat{a} \rightarrow \hat{a}$	yes
$\text{Int} \rightarrow \exists a.a \rightarrow a$ $\text{Int} \rightarrow \hat{a} \rightarrow \hat{a}$	no
$\text{Int} \rightarrow \exists a.a \rightarrow a \rightarrow a$ $\text{Int} \rightarrow \exists b.b \rightarrow \hat{a} \rightarrow \hat{a}$	no
$\exists a.a \rightarrow \text{Int} \rightarrow \exists b.b \rightarrow b$ $\exists a.a \rightarrow \text{Int} \rightarrow \hat{b} \rightarrow \hat{b}$	no

## Soundness

If the **algorithmic** system infers type  $\sigma'$  for  $e$ , the **declarative** system can infer any solution  $\sigma$  of  $\sigma'$ .

## Completeness?

If the **declarative** system can infer type  $\sigma$  for  $e$ , the **algorithmic** system infers  $\sigma'$  **compatible** with  $\sigma$ .

Table

Types	solution	compatible
$\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$ $\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$	yes	yes
$\text{Int} \rightarrow \exists a. a \rightarrow a$ $\text{Int} \rightarrow \exists a. a \rightarrow a$	yes	yes
$\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$ $\text{Int} \rightarrow \hat{a} \rightarrow \hat{a}$	yes	yes
$\text{Int} \rightarrow (\text{Int} \rightarrow a) \rightarrow (\text{Int} \rightarrow a)$ $\text{Int} \rightarrow \hat{a} \rightarrow \hat{a}$	yes	yes
$\text{Int} \rightarrow \exists a. a \rightarrow a$ $\text{Int} \rightarrow \hat{a} \rightarrow \hat{a}$	no	yes
$\text{Int} \rightarrow \exists a. a \rightarrow a \rightarrow a$ $\text{Int} \rightarrow \exists b. b \rightarrow \hat{a} \rightarrow \hat{a}$	no	yes
$\exists a. a \rightarrow \text{Int} \rightarrow \exists b. b \rightarrow b$ $\exists a. a \rightarrow \text{Int} \rightarrow \hat{b} \rightarrow \hat{b}$	no	yes
$[e : \exists a. a \rightarrow \text{Int} \rightarrow \exists b. b \rightarrow b]$ $[e : \exists a. a \rightarrow \text{Int} \rightarrow \hat{b} \rightarrow \hat{b}]$	no	no

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$\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$ $\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$	yes	yes
$\text{Int} \rightarrow \exists a.a \rightarrow a$ $\text{Int} \rightarrow \exists a.a \rightarrow a$	yes	yes
$\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$ $\text{Int} \rightarrow \hat{a} \rightarrow \hat{a}$	yes	yes
$\text{Int} \rightarrow (\text{Int} \rightarrow a) \rightarrow (\text{Int} \rightarrow a)$ $\text{Int} \rightarrow \hat{a} \rightarrow \hat{a}$	yes	yes
<del><math>\text{Int} \rightarrow \exists a.a \rightarrow a</math></del> $\text{Int} \rightarrow \hat{a} \rightarrow \hat{a}$	no	yes
<del><math>\text{Int} \rightarrow \exists a.a \rightarrow a \rightarrow a</math></del> $\text{Int} \rightarrow \exists b.b \rightarrow \hat{a} \rightarrow \hat{a}$	no	yes
<del><math>\exists a.a \rightarrow \text{Int} \rightarrow \exists b.b \rightarrow b</math></del> $\exists a.a \rightarrow \text{Int} \rightarrow \hat{b} \rightarrow \hat{b}$	no	yes
<del><math>[e : \exists a.a \rightarrow \text{Int} \rightarrow \exists b.b \rightarrow b]</math></del> $[e : \exists a.a \rightarrow \text{Int} \rightarrow \hat{b} \rightarrow \hat{b}]$	no	no

Predicate  $\min_{\exists}(\Gamma \vdash e \Rightarrow \sigma) \triangleq \forall \sigma'. \Gamma \vdash e \Rightarrow \sigma' \implies |\exists(\sigma)| \leq |\exists(\sigma')|$

Key Rule D-I-ABS1

$$\frac{\begin{array}{l} \Gamma, x : \tau \vdash e \Rightarrow_1 \sigma \\ \Gamma, x : \tau \vdash \sigma \rightsquigarrow_{\forall} \epsilon \\ \bar{a} \text{ fresh} \quad \epsilon' = [\bar{a}/[\epsilon]_x]\epsilon \\ \min_{\exists}(\Gamma \vdash \lambda x. e \Rightarrow \tau \rightarrow \exists \bar{a}. \epsilon') \end{array}}{\Gamma \vdash \lambda x. e \Rightarrow_1 \tau \rightarrow \exists \bar{a}. \epsilon'}$$

Completeness If the restricted declarative system (1) can infer  $\sigma$  for  $e$ , the algorithmic system infers  $\sigma'$  such that  $\sigma$  is a solution of  $\sigma'$ .

Table

Types	solution	compatible
$\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$ $\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$	yes	yes
$\text{Int} \rightarrow \exists a.a \rightarrow a$ $\text{Int} \rightarrow \exists a.a \rightarrow a$	yes	yes
$\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$ $\text{Int} \rightarrow \hat{a} \rightarrow \hat{a}$	yes	yes
$\text{Int} \rightarrow (\text{Int} \rightarrow a) \rightarrow (\text{Int} \rightarrow a)$ $\text{Int} \rightarrow \hat{a} \rightarrow \hat{a}$	yes	yes
$\text{Int} \rightarrow \exists a.a \rightarrow a$ $\text{Int} \rightarrow \hat{a} \rightarrow \hat{a}$	no	yes
$\text{Int} \rightarrow \exists a.a \rightarrow a \rightarrow a$ $\text{Int} \rightarrow \exists b.b \rightarrow \hat{a} \rightarrow \hat{a}$	no	yes
$\exists a.a \rightarrow \text{Int} \rightarrow \exists b.b \rightarrow b$ $\exists a.a \rightarrow \text{Int} \rightarrow \hat{b} \rightarrow \hat{b}$	no	yes
<del><math>[e : \exists a.a \rightarrow \text{Int} \rightarrow \exists b.b \rightarrow b]</math> <math>[e : \exists a.a \rightarrow \text{Int} \rightarrow \hat{b} \rightarrow \hat{b}]</math></del>	no	no

Predicate  $\min_{\exists}(\Gamma \vdash e \Rightarrow \sigma) \triangleq \forall \sigma'. \Gamma \vdash e \Rightarrow \sigma' \implies |\exists(\sigma)| \leq |\exists(\sigma')|$

Key Rule

D-I-APP2

$$\frac{\begin{array}{c} \Gamma \vdash e \Rightarrow_2 \sigma \\ \min_{\exists}(\Gamma \vdash e \Rightarrow \sigma) \\ \Gamma \vdash e : \sigma \rightsquigarrow \sigma_1 \rightarrow \sigma_2 \\ \Gamma \vdash e_1 \Leftarrow_2 \sigma_1 \end{array}}{\Gamma \vdash e e_1 \Rightarrow_2 \sigma_2}$$

D-C-SUB2

$$\frac{\begin{array}{c} \Gamma \vdash e \Rightarrow_2 \sigma \\ \min_{\exists}(\Gamma \vdash e \Rightarrow \sigma) \\ \Gamma \vdash e : \sigma <: \rho \end{array}}{\Gamma \vdash e \Leftarrow_2 \rho}$$

Completeness

If the restricted declarative system (2) can infer  $\sigma$  for  $e$ , the algorithmic system infers type  $\sigma'$  compatible with  $\sigma$ .

## This Work



- A **declarative** type system
  - strong existential types
  - higher-rank polymorphism
  - subtyping
- An **algorithmic** type system
  - unification and promotion
  - sound but incomplete w.r.t. the decl. system
  - complete w.r.t. 2 fragments of the decl. system
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- Haskell implementation

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TX  
ESOP 2026

## Bidirectional Type Checking for Existential Types with Higher-Rank Polymorphism

Hasti Toossi  and Ningning Xie University of Toronto, Toronto, Canada  
{hasti2c,ningningxie}@cs.toronto.edu

**Abstract.** This paper presents a new type system that combines strong existential types, higher-rank polymorphism, and polymorphic subtyping. We begin by presenting a declarative type system, and contributing several metatheoretic results. We then detail an algorithmic type inference system featuring bidirectional type checking. We prove that the algorithm is sound. Moreover, we propose two design variants of the declarative system with respect to which the algorithmic system is complete. This work can serve as a basis for further studies on strong existentials, their corresponding type inference algorithms, and their combinations with higher-rank polymorphism and subtyping.